B.Math. (Hons.) IInd year
Second semestral exam 2017
Topology - Instructor : B.Sury
Maximum Marks : 50

Q 1.
Define the lower limit topology on $\mathbf{R}$. For this space $\mathbf{R}_l$, decide which of the following properties hold, giving a brief/one-line argument:
(a) connectedness, (b) compactness, (c) second countability, (d) normality.

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Q 2.(5+5)
(i) Let A be a nonempty subset of a topological space. Show that the interior of the closure of A equals the interior of the closure of the interior of the closure of A.
(ii) Consider $\mathbf{R} \times \mathbf{R}$ under the order topology coming from the dictionary order. Prove that as a topological space, this has the product topology on $X \times Y$ where X is $\mathbf{R}$ with the Euclidean topology and Y is $\mathbf{R}$ with the discrete topology.

OR

Q 2.(5+5)
(i) Prove that every ordered topology is Hausdorff.
(ii) If $f : X \to Y$ is a continuous map from a compact space to a Hausdorff space, show that f is a closed map.

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Q 3.(3+4+3)
(i) Prove that $[0,1] \times [0,1]$ in the dictionary order topology is locally connected but not locally path-connected.
(ii) Show that the topology of $[0,1] \times [0,1]$ as a subspace of $\mathbf{R} \times \mathbf{R}$ under the dictionary order is strictly finer than the dictionary order topology on $[0,1] \times [0,1]$.
(iii) Let $X = \cup_n C_n$, where C_n's are connected subsets such that $C_n \cap C_{n+1} \neq \emptyset$. Show that X is connected.

OR

Q 3. (3+3+4)
(i) Define a regular space and a normal space.
(ii) Prove that a metric space is normal.
(iii) Prove that $\mathbf{R}$ with the K-topology is Hausdorff but is not regular.

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Q 4. (3+5+2)
(i) State the Tietze extension theorem. Provide an example to show that its conclusion may not be true if the subset considered is not closed.
(ii) Let X be a complete metric space. If $\{C_n\}_n$ is a nested sequence of nonempty closed subsets of X such that diam $(C_n) \to 0$, prove that $\bigcap_n C_n$ is non-empty.
(iii) Show that the metric space $\mathbf{N}$ with $d(m,n) = 1 + \frac{1}{m+n}$ for $m \neq n$ is complete and that the sets $\{n, n+1, \cdots\}$ are closed.

OR

Q 4. (2+3+5)
(i) Show that the set of irrational numbers is a Baire space.
(ii) Prove that a topological space is normal if, and only if, for each closed subset C and open set $U \supset C$, there exists an open set V such that

$$C \subset V \subset \bar{V} \subset U.$$

(iii) Let X be a compact Hausdorff space and $f : X \to X$ be continuous. Consider the sequence of spaces $X, f(X), f(f(X)), \cdots$ etc. That is, $X_1 = X$, and $X_{n+1} = f(X_n)$ for all $n > 0$. Prove that $X_0 := \bigcap_n X_n \neq \emptyset$ and that $f(X_0) = X_0$.

Q 5. (5+5)
(i) Let X be a topological space and let $f, g : X \to \mathbf{C} \setminus \{0\}$ be continuous functions such that $|f(x) - g(x)| < |f(x)|$ for all $x \in X$. Prove that f and g are homotopic.
(ii) Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be continuous and suppose $||f(x) - x|| < 1$ for all x. Use Brouwer's fixed point theorem to show that f is surjective.

OR

Q 5. (6+4)
(i) Define a covering map $p : X \to Y$. Prove that if X, Y are connected, then each fibre has the same cardinality.
(ii) If X is simply-connected, prove that any two paths with the same starting and end points are homotopic.